Lectures 18 & 19

Plan:

1) ~~briefly recap matroid polytope~~
2) ~~Matroid intersection activity~~
3) largest common independent subset

Matroid intersection

- Matroids very nice b/c greedy works. given $c: E \to \mathbb{R}$, $\max_{S \in \mathcal{I}} c(S) = \sum_{e \in S} c(e)$
- But greedy doesn't work for lots of problems,

  e.g. ▷ max matching,

  ▷ max stable set in graph.

$\Rightarrow$ matroids very limited!

$|2^E| = 2^{|E|}$

A.K.A powerset.

matroid intersection much richer.

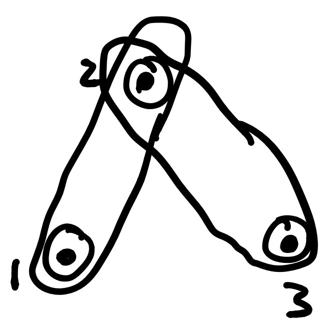
2^E = set of subsets of E e.g. $2^{\{1,2\}} = \{\{1,2\}, \{1\}, \{2\}, \emptyset\}$.

Def If $M_1 = (E, \mathcal{I}_1)$, $M_2 = (E, \mathcal{I}_2)$ matroids on common ground set E, their **intersection** is just

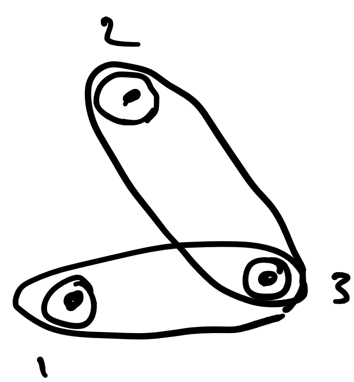
$$\mathcal{I}_1 \cap \mathcal{I}_2 \subseteq 2^E, \quad (\mathcal{I}_1, \mathcal{I}_2 \subseteq 2^E)$$

i.e. the sets indep. in M_1 & M_2.

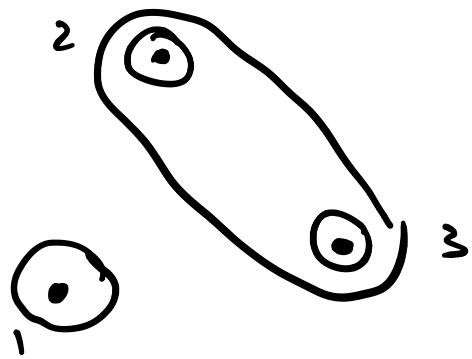
E.g. $E = \{1,2,3\}$

$\mathcal{I}_1 =$ 

$+ \emptyset$

$\mathcal{I}_2 =$ 

$+ \emptyset$

$I_1 \cap I_2 =$ 

$+ \emptyset.$

- Activity: lots of examples!
- We'll show how to find largest common independent set efficently! (Next time, probably).

e.g. largest bipartite matching

$$\max_{S \in I_1 \cap I_2} |S|$$

Largest Common indep. set

- we give a min-max characterization for L.C.I.S.
  i.e. "duality" result
- allows us to prove:

- Let M_1, M_2 be matroids w/ rank functions r_1 r_2

- Let $S \in \mathcal{I}_1 \cap \mathcal{I}_2$ be common indep. set, $U \subseteq E$ any subset of ground set.

Then
$$
\begin{aligned}
|S| &= |S \cap U| + |S \setminus U| \\
&= r_1(S \cap U) + r_2(S \setminus U) \\
&\le r_1(U) + r_2(E \setminus U).
\end{aligned}
$$

$S \cap U \in \mathcal{I}_1 \,\&\, \mathcal{I}_2$

$S \setminus U \in \mathcal{I}_2 \,\&\, \mathcal{I}_1$

- max over S, min over U:

$$\max_{S \in \mathcal{I}_1 \cap \mathcal{I}_2} |S| \le \min_{U \subseteq E} r_1(U) + r_2(E \setminus U).$$

"strong duality":

4th pset problem helps build intuition

Theorem: (Edmonds)

$$\max_{S \in \mathcal{I}_1 \cap \mathcal{I}_2} |S| = \min_{U \subseteq E} r_1(U) + r_2(E \setminus U).$$

Remark: Enough to minimize over U closed for M_1 ($closed := span(U) = U$)

- $U \leftarrow span_{M_1}(U)$ doesn't increase RHS

OR ~~similarly~~ can assume $E \setminus U$ closed in M_2

- same reason ...
- but NOT both at same time!

E.g. Special cases:

- Can show (Exercise) that orienting G w/ indegree $\le p(v)$ possible $\Leftrightarrow \forall S \subseteq V \quad |E(S)| \le \sum_{v \in S} p(v)$ (as in Pset 4).

- Can show $\exists$ colorful spanning tree $\Leftrightarrow$ deleting any c colors produces $\le c+1$ connected components.

Proof of Theorem

- proof is "primal-dual", i.e. at each step either increase $|S|$ or terminate & output U with
$$|S| = r_1(U) + r_2(E \setminus U).$$

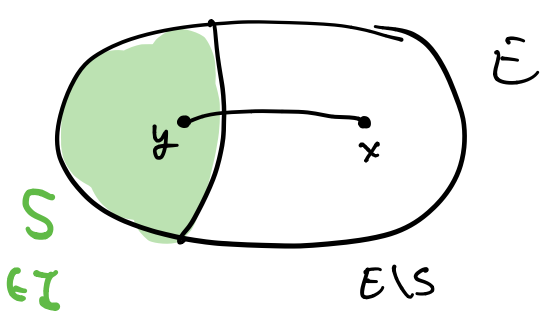
- Uses "directed exchange graph".
work up to full definition.

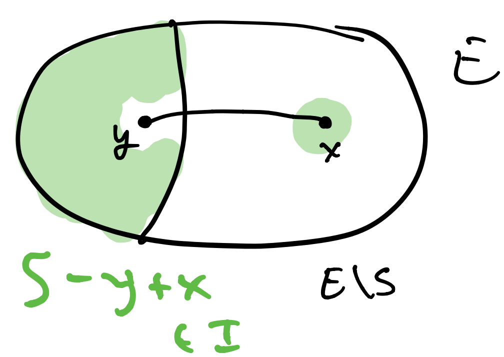
first, undirected: from one matroid $M = (E, \mathcal{I})$

Def Given $S \in \mathcal{I}$, (undirected) exchange graph

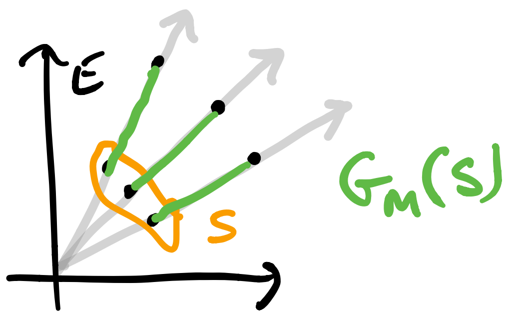
$G_M(S) :=$
- bipartite graph, vertices = ground set E
- partition S, $E \setminus S$
- (y, x) edge if $S - y + x \in \mathcal{I}$
  $y \in S$, $x \in E \setminus S$

matching example
ground set is edges of graph
2 partition matroids





e.g. a linear matroid

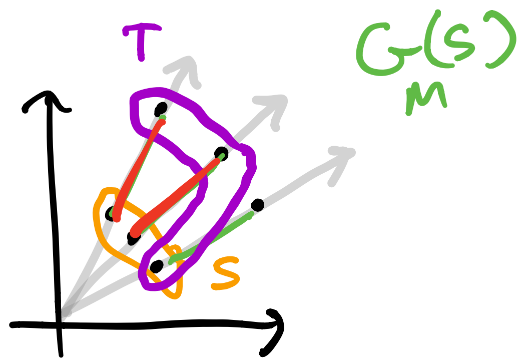


equivalent:
(y,x) edge $\iff$
$x \notin \mathrm{span}(S-y)$.

- Arises in "matroid sampling", sampling a uniformly random base of matroid. (Anari et. al.)
- For us, useful for the following reason:

Lemma: Let $S, T \in \mathcal{I}$ $|S| = |T|$. Then $G_M(S)$ has a perfect matching between $S \setminus T$ and $T \setminus S$.

eg.



proof: Exercise (repeatedly apply exchange axiom).

and a partial converse:

Lemma: Let $S \in \mathcal{I}$, $T \subseteq E$ s.t. $|S| = |T|$.
Suppose $G_M(S)$ has unique perfect matching between $S \setminus T$ & $T \setminus S$.

$\boxed{\text{Then } T \in \mathcal{I}.}$

Proof: Let N be unique matching.

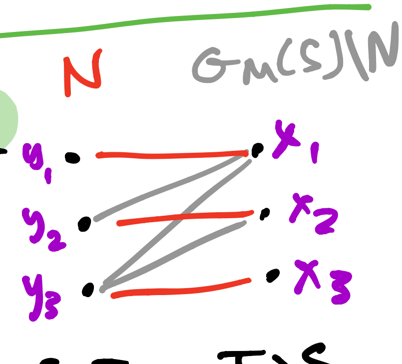
- ordering result:

Claim: Can order

$S \setminus T = \{y_1, \dots, y_k\}$

$T \setminus S = \{x_1, \dots, x_k\}$

e.g.



so that $N = \{(y_1, x_1), \dots, (y_k, x_k)\}$

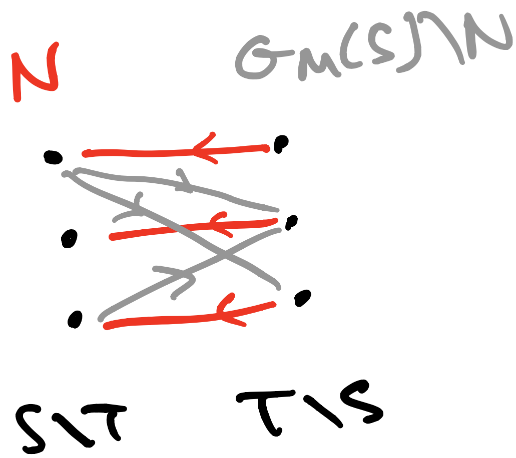
and $(y_i, x_j) \in G_M(S)$ for $i < j$.

- proof is just about graphs (not matroids).

proof of claim:

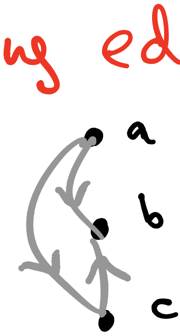
- Ignore edges not between $S \setminus T$, $T \setminus S$.
- Orient N from $T \to S$
- Others $S \to T$

e.g.



- Contract along edges of N. (remove loops).

e.g.



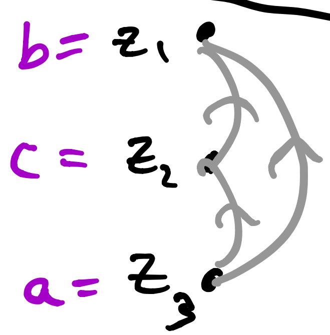
- Get <u>acyclic</u> directed graph

(else get alternating cycle in $G_M(S)$ w.r.t. N, contradicts uniqueness of N).

- <u>Topologically order</u> vertices of the "contracted graph so that all edges point backwards. (possible b/c acyclic.)

$z_1 \ldots z_k$.

here



- Let $x_i \in T\setminus S$, $y_i \in S\setminus T$ be the vertices we contracted to get z_i. $\square$

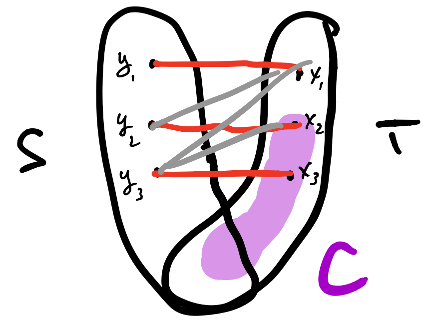
(can now forget about contracted digraph/top. ordering).

Now, for contradiction: suppose $T \not\subseteq \mathcal{I}$.

minimal dependent set. $\forall x \in C$, $x \in \text{span}(C - x)$.

- then T contains a circuit C.

e.g.

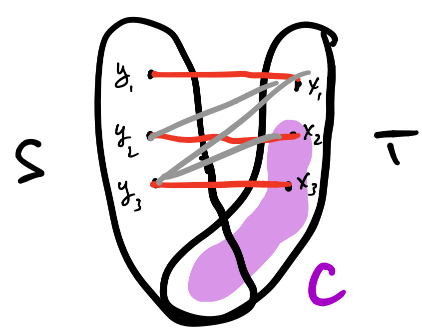


- then C intersects $T \setminus S$

(else $C \subseteq S$, contradict $S \in \mathcal{I}$).

- Let x_i be first element in C (under ordering).

e.g.



here $i = 2$.

- Now we'll find that $x_i \in \mathrm{span}(S - y_i)$
contradicts $(y_i, x_i) \in G_M(S)$!

(by def. of $G_M(S)$.).

- To show this, observe

$$\forall x \in C - x_i \,,\; x \in \mathrm{span}(S - y_i)$$

b/c $(y_i, x) \notin G_M(S)$ by ordering.
(x comes after x_i in ordering).

$$\Rightarrow \mathrm{span}(S - y_i) \supseteq \mathrm{span}(C - x_i) \ni x_i \; *$$

by properties of span.

because C circuit.

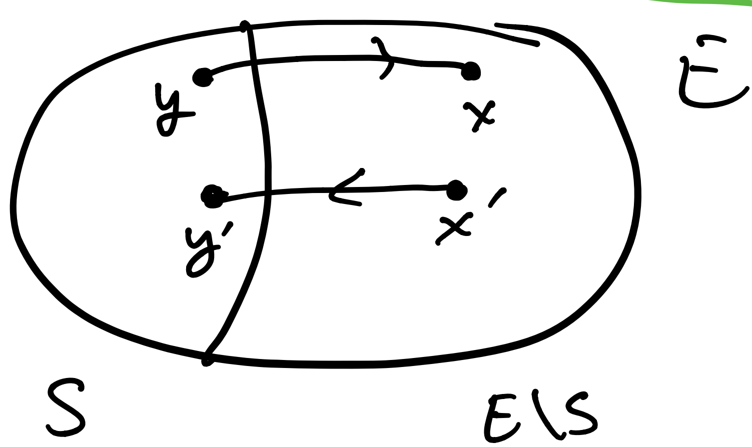
$\square$

- now generalize to <u>directed exchange graph</u> from <u>two</u> matroids $M_1 = (E, \mathcal{I}_1)$, $M_2 = (E, \mathcal{I}_2)$

<u>**Def**</u> For $S \in \mathcal{I}_1 \cap \mathcal{I}_2$, (directed) <u>**exchange graph**</u>

$D_{M_1, M_2}(S) :=$
- directed bipartite graph
- parts S, $E \setminus S$
- $(\underset{S}{y}, \underset{E \setminus S}{x})$ edge if $S - y + x \in \mathcal{I}_1$
- $(\underset{E \setminus S}{x}, \underset{S}{y})$ edge if $S - y + x \in \mathcal{I}_2$

<u>**Picture**</u>:



here
$S - y + x \in \mathcal{I}_1$, $S - y' + x' \in \mathcal{I}_2$

- <u>Note</u>: $G_{M_1}(S)$, $G_{M_2}(S)$ are subgraphs rightwards edges   leftwards edges.

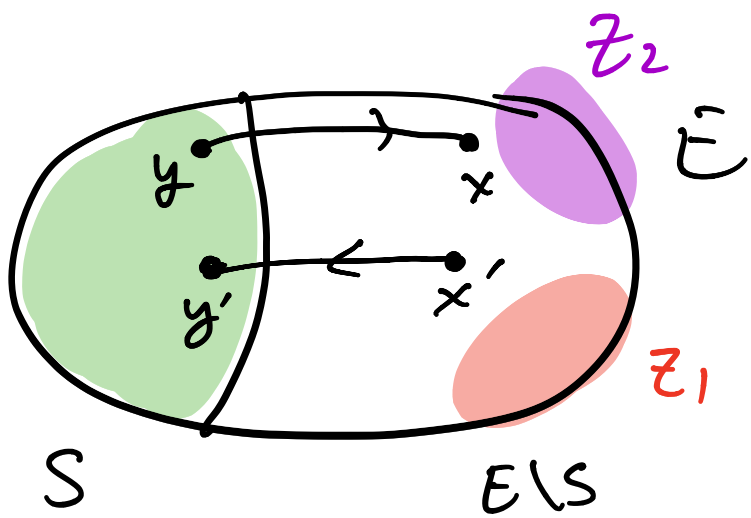
- **Also define:**

"sources" $Z_1 := \{x \notin S : S + x \in \mathcal{I}_1\}$

i.e $S+x$ independent in M_1

"sinks" $Z_2 := \{x \notin S : S + x \in \mathcal{I}_2\}$.

e.g.



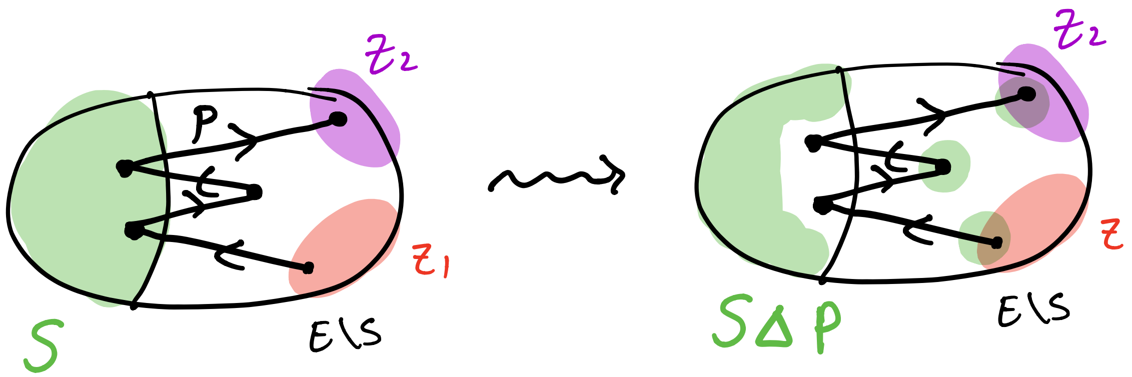
Algorithm Initialize $S = \emptyset$.

- **Repeat:** (until termination)
  - Compute $D_{M_1 M_2}(S)$
  - if $\exists$ directed path from sources Z_1 to sinks Z_2 in $D_{M_1 M_2}(S)$:

▷ $P :=$ a <u>shortest</u> such path.

▷ Replace $S \leftarrow S \Delta P$

($P :=$ vertices on path).



▷ else: (i.e. no path)

▷ return

$U = \{ z \in E : \text{sinks } Z_2 \text{ are reachable from } z \text{ in } D_{M_1 M_2}(S) \}$.

<u>Correctness:</u>

- <u>Claim 1</u>: S remains in $I_1 \cap I_2$
- <u>Claim 2</u>: $|S| = r_1(U) + r_2(E \setminus U)$ at termination.

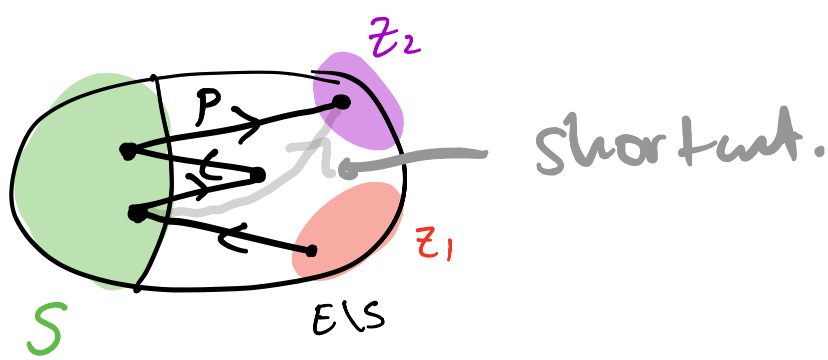
$\# \text{steps} \le O(|E|^3)$

Proof of Claim 1:

want to show $S \Delta P \in I_1 \cap I_2$

- Recall P shortest path; in particular no shortcuts

e.g.



- Enough to show: P has no shortcuts $\Rightarrow S \Delta P \in I_1 \cap I_2$.

- We first show $S \Delta P \in I_1$

- To do this, define new matroid $M_1' = (E', I_1')$ by add new element t to E, and defin

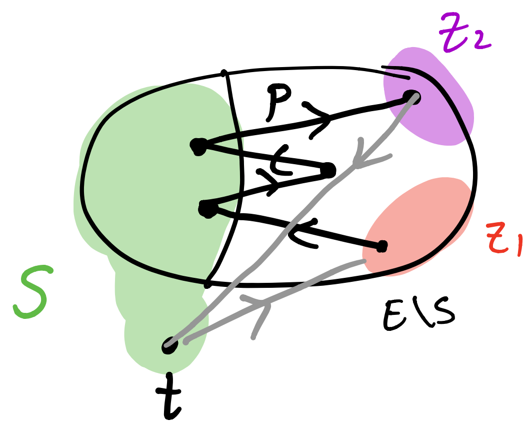
$E' = E + t$ and $\mathcal{I}' = \{ J : J - t \in \mathcal{I}_1 \}$

i.e. add t & make it "independent from everything in E". $\mathcal{I}' = \{ R, R+t : R \in \mathcal{I}_1 \}$.

- Define M_2' analogously (using same t)

consider $D_{M_1', M_2'}(S+t)$.

e.g.



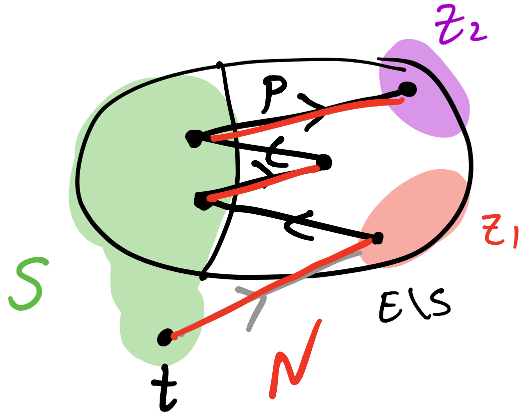
$D_{M_1' M_2'}(S+t)$

- Note $D_{M_1' M_2'}(S+t)$ is just $D_{M_1, M_2}(S)$ plus all edges $t \to z_1$, $t \leftarrow z_2$

- View $G_{M_1'}(S+t)$ as a subgraph of $D_{M_1', M_2'}(S+t)$ undirected

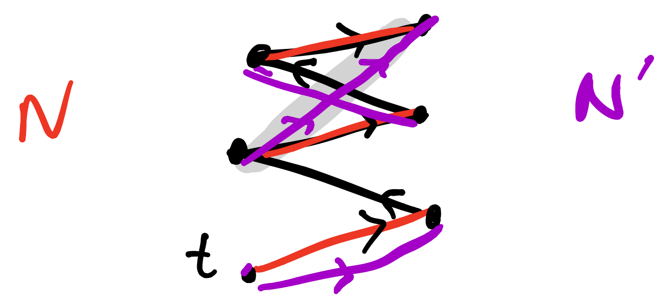
- Observe $G_{M_1'}(S+t)$ contains a p.m. N between $S \cap P + t$ & $P \setminus S$.

e.g.



(include edge $t \to$ start of P & all edges of P starting in S).

- And N is unique by no shortcut property.
  between $S \cap P + t$, $P \setminus S$.



$(G_{M_i}(S+t)$ is directed rightwards,
So $N' \neq N$ perfect matching yields shortcut$)$

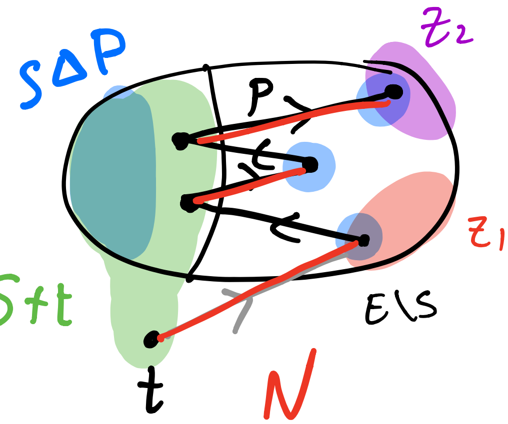
$G_{M_i}(S+t)$

- Unique perfect matching lemma

$\Rightarrow S \Delta P \in I_1'$

$\Rightarrow S \Delta P \in I_1$ ✓ (by def of M_i').

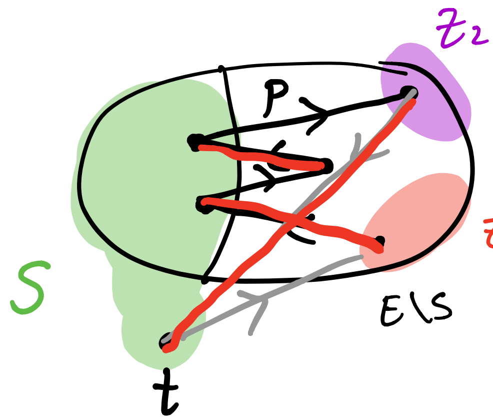
e.g.



- To show $S \Delta P \in I_2$ instead

find matching in $G_{M_2'}(S+t)$
use edge from last vertex in P to t.

e.g.



proof similar

- finishes proof of Claim 1. $\square$.

Proof of Claim 2:

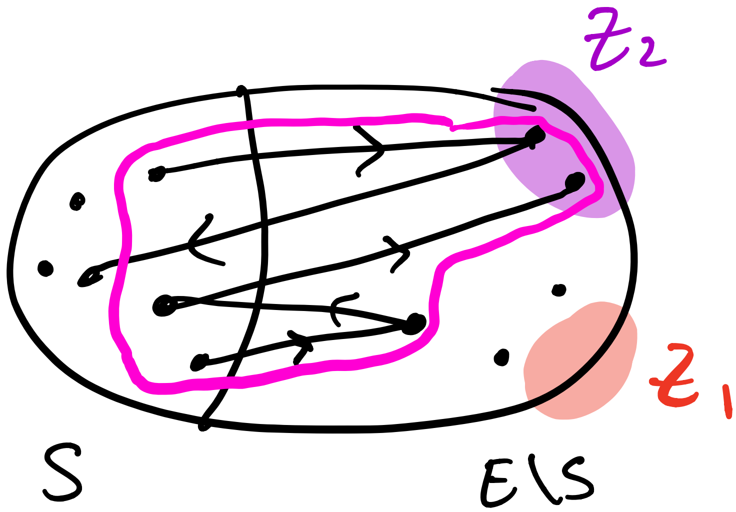
i.e. no $z_1 \to z_2$ path

- Want to show at termination,

$$|S| = r_1(U) + r_2(E \setminus U)$$

where U = everything from which some vertex of Z_2 is reachable.

e.g.



- First note $Z_2 \subseteq U$ and $Z_1 \cap U = \emptyset$

  else algo. not done.

- Enough to show $r_1(U) = |S \cap U|$

  & $r_2(E\setminus U) = |S\setminus U|$ ☆

  (then $|S| = |S\cap U| + |S\setminus U|$

  $= r_1(U) + r_2(E\setminus U)$)

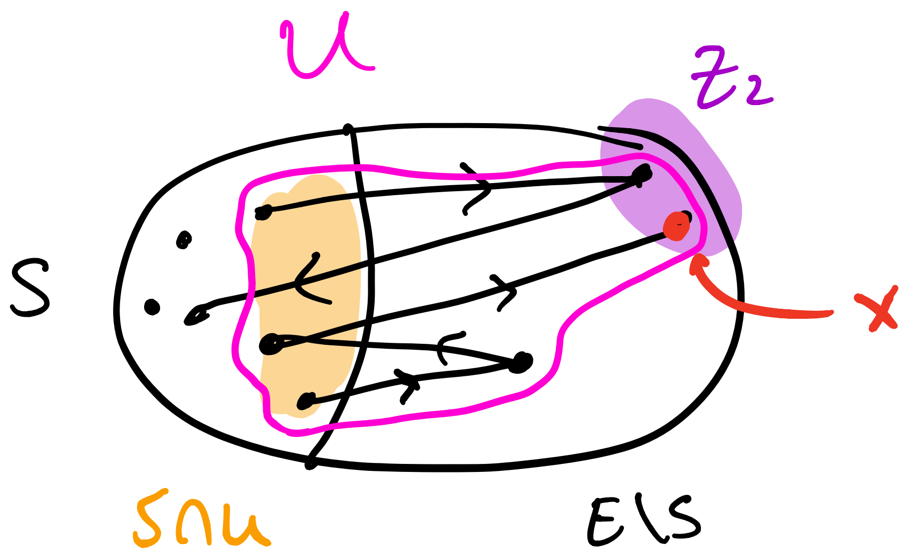
- Suppose $r_1(U) \neq |S \cap U|$.

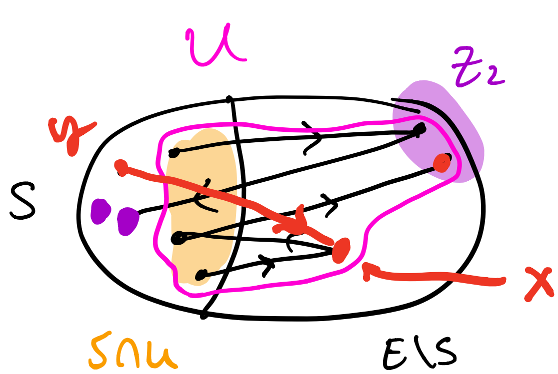
- $S \cap U \subseteq U$, S indep in M_1 $\Rightarrow$ $\underset{\text{indep in } M_1}{S \cap U}$

  $\Rightarrow |S \cap U| < r_1(U)$.

  $\Rightarrow \exists x \in U \setminus S$ s.t. $(S \cap U) + x \in \mathcal{I}_1$

  exchange axiom.

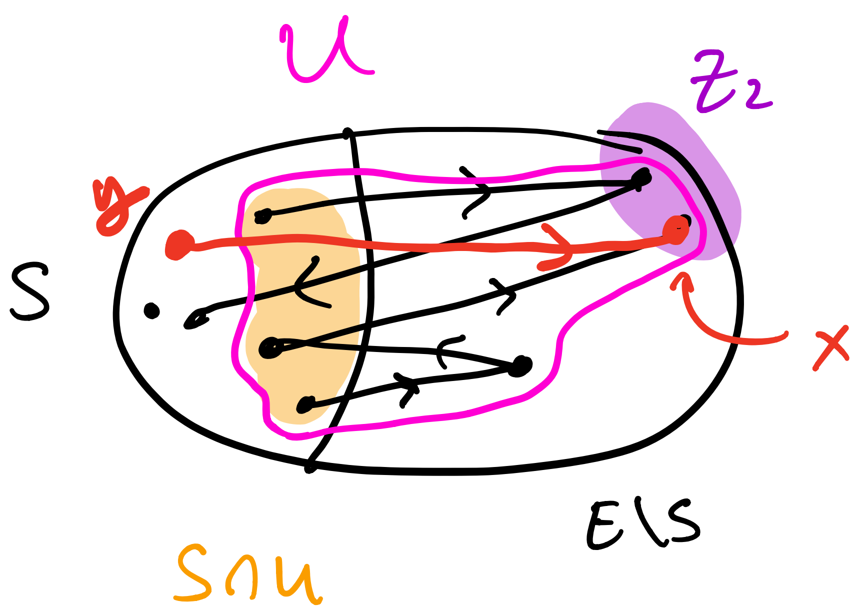




- $S \in \mathcal{I}_2 \Rightarrow$ Can add elts of $S \setminus U$ to $(S \cap U) + x$ until we obtain a set $S + x - y \in \mathcal{I}_1$ for $y \in S \setminus U$.

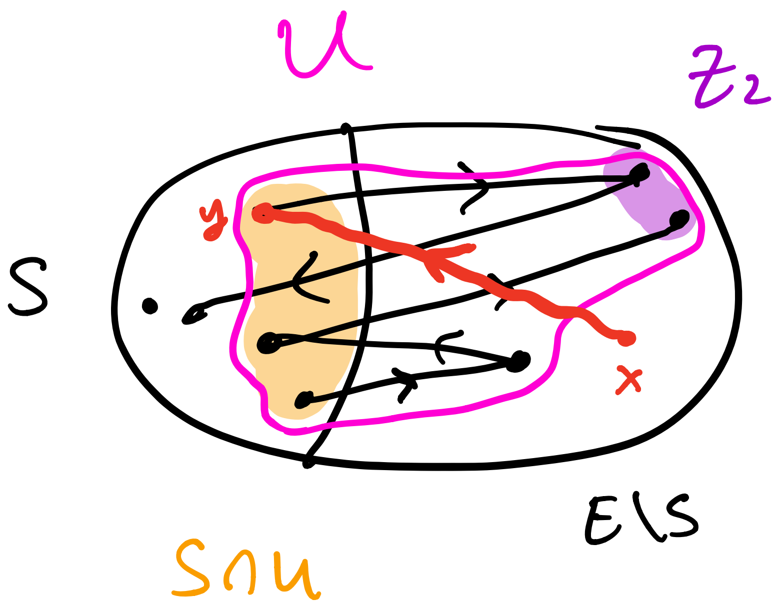
(repeatedly apply exchange axiom. First to $S \cap U + x$, S ....)

- But then (y, x) is in $D_{M_1, M_2}(S)$.
$\Rightarrow y \in U$; contradicts $y \in S \setminus U$.



- Case $r_2(E\setminus U) \neq |S\setminus U|$

similar; contradiction looks like



left as exercise. $\square$.